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## The Twistor Space of Distributions

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## THE TWISTOR SPACE OF DISTRIBUTIONS

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**1. Introduction.** In this paper a naturally defined kind of distributions on the twistor space of distributions over some manifold  $N$  is established. It is defined by an ordered pair  $(D, \omega)$  of a distribution  $D$  on the base manifold  $N$  and a linear connection  $\omega$  on it. Concerning integrability of this distribution (denoted by  $D^\omega$ ) it is shown that cases  $\dim D^\omega = 2$  and  $\dim D^\omega = \dim N$  are distinguished. Nevertheless  $D^\omega$  is integrable very rare in any case (see theorems 2.2 and 2.4). Moreover consequences of induced action of a gauge transformation on this distribution are also considered. One works in the  $C^\infty$  category.

**2. Constructions.** Let  $\pi_L: L(N) \rightarrow N$  be the principal  $GL(n; \mathbb{R})$  bundle of linear frames over an  $n = p + q$ -dimensional manifold  $N$ . The twistor space of  $p$ -dimensional distributions over  $N$  is constructed as follows: for the subgroup

$$(2.1) \quad G = \left\{ \begin{pmatrix} A & B \\ O & C \end{pmatrix} \in GL(n; \mathbb{R}) \mid A \in GL(p; \mathbb{R}), C \in GL(q; \mathbb{R}) \right\}$$

any reduced  $G$ -subbundle  $Q \subset L(N)$  corresponds to some  $p$ -dimensional distribution  $D \subset TN$  on the manifold  $N$ , [3]. There exists a 1:1 correspondence between such subbundles and the set of sections of a fiber bundle associated with  $L(N)$  whose standard fiber is the space  $GL(n; \mathbb{R})/G$ . This fiber bundle can be naturally identified with the quotient  $L(N)/G$ ; and this quotient represents the twistor space  $Z$  of  $p$ -dimensional distributions over manifold  $N$ . This generalized framework is considered in [1] and [4] in another setting.

Fibrations are exposed by

$$(2.2) \quad L(N) \xrightarrow{\pi_L} Z = L(N)/G \xrightarrow{\pi} N.$$

Thus any  $p$ -dimensional distribution  $D \subset TN$  on  $N$  uniquely determines a global section (also denoted by)  $D: N \rightarrow Z$  which further defines a reduced  $G$ -subbundle  $Q := \pi_Z^{-1}(D(N))$ . By definition some linear connection  $\omega$

on  $L(N)$  is reducible to  $Q$  iff its horizontal distribution  $H^\omega \subset TL(N)$  is tangent to  $Q$ . Equivalently,  $\omega$  is reducible to  $Q$  iff

$$(2.3) \quad (\pi_Z)_*(H^\omega|_Q) = D_*(TN)$$

holds. In the same manner a horizontal distribution on the space  $Z$  is defined by

$$(2.4) \quad (\pi_Z)_*(H^\omega) \subset TZ.$$

Here  $\omega$  need not be reducible to  $Q$ . Consequently if  $D^H \subset H^\omega$  is the horizontal lift of a distribution  $D \subset TN$  defined by the connection  $\omega$  on  $L(N)$ , then a distribution  $D^\omega \subset TZ$  is defined on the space  $Z$  as

$$(2.5) \quad D^\omega := (\pi_Z)_*(D^H).$$

Before considering the question of integrability of this distribution first will be proved the following very useful

**Lemma 2.1.** *Let  $B_i$  be the standard horizontal vector fields on bundle  $L(N)$  defined by a connection  $\omega$  and basis vectors  $u_i \in \mathbb{R}^n$  ( $i = 1, \dots, n$ ), [2]. Denote by  $\Omega$  and  $T$  the curvature and the torsion of  $\omega$  respectively (viewed on  $L(N)$ ). Then the following commutator formula holds.*

$$(2.6) \quad [B_i, B_j] = -T^k(B_i, B_j)B_k - \Omega_i^k(B_i, B_j)E_k^l.$$

Here  $(E_k^l)$  is the standard basis in  $\mathfrak{gl}(n; \mathbb{R})$ , and one identifies (i.e., uses the same letter) elements from  $\mathfrak{gl}(n; \mathbb{R})$  and corresponding fundamental fields on  $L(N)$  (a convention is  $v_1 \wedge v_2 := v_1 \otimes v_2 - v_2 \otimes v_1$ ).

*Proof.* Let  $M$  be a manifold and  $(e_\alpha)$  and  $(\varepsilon^\alpha)$  some frame with corresponding coframe on  $M$  respectively. Then holds the following relations ([5, p. 26])

$$(2.7) \quad \begin{aligned} [e_\alpha, e_\beta] &= C_{\alpha\beta}^\gamma e_\gamma, \\ d\varepsilon^\gamma &= -\frac{1}{2}C_{\alpha\beta}^\gamma \varepsilon^\alpha \wedge \varepsilon^\beta. \end{aligned}$$

Here  $C$ -symbols are defined by the choice of frame  $(e_\alpha)$ . Let

$$(2.8) \quad \begin{aligned} d\theta^i &= -\omega_j^i \wedge \theta^j + \frac{1}{2}T_{jk}^i \theta^j \wedge \theta^k \\ d\omega_j^i &= -\omega_k^i \wedge \omega_j^k + \frac{1}{2}\Omega_{jkl}^i \theta^k \wedge \theta^l \end{aligned}$$

be the structure equations of the connection  $\omega$ . Here  $\Omega_{jkl}^i := \Omega_j^i(B_k, B_l)$ ,  $T_{jk}^i := T^i(B_j, B_k)$ , and  $(\theta^i)$  are components of the canonical 1-form on  $L(N)$ . Because  $(\theta^i; \omega_j^i)$  is a (global) coframe on  $L(N)$  which corresponds to the frame  $(B_i; E_j^i)$ , by comparing (2.7) and (2.8) one obtains (2.6).

Let  $X^\omega, Y^\omega$  and  $[X, Y]^\omega$  be horizontal lifts to  $L(N)$  of vector fields  $X, Y$  and  $[X, Y]$  on  $N$  respectively which are defined by the connection  $\omega$  on  $L(N)$ . Then by using (2.6) one finds the following relation

$$(2.9) \quad [X^\omega, Y^\omega] = [X, Y]^\omega - T^k(X^\omega, Y^\omega)B_k - \Omega(X^\omega, Y^\omega).$$

By this formula one is able to prove the following

**Theorem 2.2.** *Let  $D \subset TN$  be a  $p$ -dimensional foliation on a manifold  $N$  endowed with a torsionfree connection  $\omega$  by which the distribution  $D^\omega \subset TZ$  introduced in (2.5) is defined. Suppose that vector fields  $Y_1$  and  $Y_2$  take value in  $D$  while  $Y_3$  is any vector field on  $N$ . Then the distribution  $D^\omega$  will be integrable if*

$$(2.10) \quad R(Y_1, Y_2)Y_3 = fY_3$$

*holds. Here  $R$  is the curvature tensor of  $\omega$  while  $f$  is a function on  $N$  which does not depend on  $Y_3$ .*

*Proof.* From (2.9) it follows that  $D^\omega$  will be integrable iff  $\Omega(X^\omega, Y^\omega)$  takes value in  $\mathfrak{S}$ , the Lie algebra of  $G$ , when  $X$  and  $Y$  take value in  $D$ . But fields  $X^\omega$  and  $Y^\omega$  are right  $\text{GL}(n; \mathbb{R})$ -invariant, consequently

$$(2.11) \quad \Omega_{pg}(X^\omega, Y^\omega) = \text{Ad}(g^{-1})\Omega_p(X^\omega, Y^\omega)$$

holds for all  $p \in L(N)$ ,  $g \in \text{GL}(n; \mathbb{R})$ . Thus in order that  $D^\omega$  is integrable  $\Omega(X^\omega, Y^\omega)$  must take value in an ideal of  $\mathfrak{gl}(n; \mathbb{R})$  which is contained in  $\mathfrak{S}$ , e.g.,

$$(2.12) \quad \Omega(X^\omega, Y^\omega) = \hat{f}I$$

can hold. where  $\hat{f} \in C^\infty(L(N))$ . This condition is expressed on the level of  $N$  in a form given in (2.10).

Because it is supposed that  $\omega$  is a torsionfree connection, the first Bianchi identity

$$(2.13) \quad R(Y_1, Y_2)Y_3 + R(Y_2, Y_3)Y_1 + R(Y_3, Y_1)Y_2 = 0$$

holds. Hence if in (2.10) also  $Y_3 \in D$  and moreover  $Y_1, Y_2, Y_3$  are linearly independent, then from (2.13) it follows that  $f \equiv 0$ . Thus in the setting of Theorem 2.2 we have

**Corollary 2.3.** *If  $\dim D > 2$ , then  $D^\omega$  is integrable if  $\omega$  is flat.*

On the other hand, from (2.13) and (2.10) it does not follow that there are no nontrivial examples (i.e.,  $\omega$  is not flat) of integrable  $D^\omega$  in case of  $\dim D = 2$ . Important special case is  $\dim D = \dim N$ , i.e.,  $D^\omega = (\pi_Z)_* H^\omega$  from (2.4): here there is no need to impose condition that a connection  $\omega$  is torsionfree because the term  $T^k(X^\omega, Y^\omega)B_k$  in (2.9) is not troublesome at all now. Consequently the first Bianchi identity does not make restrictions so that the following holds.

**Theorem 2.4.** *The horizontal distribution  $D^\omega$  on the twistor space  $Z$  which is introduced in (2.4) is integrable if*

$$(2.14) \quad R(Y_1, Y_2)Y_3 = fY_3$$

*holds for any vector fields  $Y_i$  on  $N$ . Here the function  $f$  on  $N$  does not depend on  $Y_3$ .*

One can notice that if  $\omega$  is a torsionfree connection on  $L(N)$  which is reducible to a  $G$ -subbundle  $Q \subset L(N)$  defined by a foliation  $D \subset TN$ , then the distribution  $D^\omega|_{D(N)}$  on the submanifold  $D(N)$  is integrable.

**3. On gauge-action upon distributions.** Let  $f: L(N) \rightarrow L(N)$  be any gauge-transformation in the bundle  $L(N)$ . Because its fiber-preserving action commutes with the right action of the group  $GL(n; \mathbb{R})$  on  $L(N)$ , it follows that  $f$  induces in a natural manner a transformation (also denoted by  $f$ ) on the twistor space  $Z$ . Consequently  $f$  maps any distribution  $D: N \rightarrow Z$  into another  $fD$ . Then  $fD = D$  holds iff  $f$  preserves the  $G$ -structure  $Q \subset L(N)$  which corresponds to  $D$ . On the other hand  $f$  acts naturally upon  $TL(N)$  and further upon  $TZ$ . In this setting we have

**Theorem 3.1.** *If the subbundle  $Q \subset L(N)$  which corresponds to a distribution  $D \subset TN$  is left invariant by a gauge-transformation  $f$ , and some connection  $\omega$  on  $L(N)$  is reducible to  $Q$ , then the distribution  $D^\omega|_{D(N)} \subset TD(N)$  is preserved under the action of  $f$  upon  $TZ$ .*

*Proof.* A sufficient condition that  $f$  preserves the distribution  $D^H \subset TL(N)$  is that it preserves the connection  $\omega$ , i.e., its horizontal distribution (if  $D^H \equiv H^\omega$ , i.e.,  $D \equiv TN$ , this condition is also necessary). But if  $fD = D$  and  $\omega$  is reducible to  $Q$  then  $f^*\omega$  is also reducible to  $Q$ . Consequently by (2.3)

$$(\pi_Z)_*(H^\omega|_Q) = (\pi_Z)_*(H^{f^*\omega}|_Q)$$

where from follows the statement of the theorem.

From this Theorem it follows that if  $fQ = Q$  and  $\omega$  is reducible to  $Q$ , then

$$D^{f^*\omega}|_{D(N)} = D^\omega|_{D(N)}.$$

From this does not follows that  $D^{f^*\omega} = D^\omega$  on the whole space  $Z$ . A sufficient condition for it is, say, that  $f^*\omega - \omega$  take value in the center  $\{cI \mid c \in \mathbb{R}\}$  of  $\mathfrak{gl}(n; \mathbb{R})$ .

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